Feynman checkers: the probability to find an electron vanishes nowhere inside the light cone

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Preliminaries (1)

Proposition (Dirac equation)

For each $(x,t) \in \varepsilon\mathbb{Z}^2$, *where* $t \geqslant 2\varepsilon$, *we have*

$$a_1(x,t,m,\varepsilon) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_1(x+\varepsilon,t-\varepsilon,m,\varepsilon) + m\varepsilon\, a_2(x+\varepsilon,t-\varepsilon,m,\varepsilon));$$

$$a_2(x,t,m,\varepsilon) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_2(x-\varepsilon,t-\varepsilon,m,\varepsilon) - m\varepsilon\, a_1(x-\varepsilon,t-\varepsilon,m,\varepsilon)).$$

Proposition

For each $(x,t) \in \varepsilon\mathbb{Z}^2$, *where* $t \geqslant 2\varepsilon$, *we have*

$$a_1(x,t-\varepsilon,m,\varepsilon) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_1(x-\varepsilon,t,m,\varepsilon) - m\varepsilon\, a_2(x+\varepsilon,t,m,\varepsilon));$$

$$a_2(x,t-\varepsilon,m,\varepsilon) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_2(x+\varepsilon,t,m,\varepsilon) + m\varepsilon\, a_1(x-\varepsilon,t,m,\varepsilon)).$$

Proof.

Use the Dirac equation. □

Preliminaries (2)

Proposition (Symmetry)

For all $(x, t) \in \varepsilon\mathbb{Z}^2$ *with* $t > 0$ *we have*
$a_1(x, t, m, \varepsilon) = a_1(-x, t, m, \varepsilon)$;
$(t - x)a_2(x, t, m, \varepsilon) = (t + x - 2\varepsilon)a_2(2\varepsilon - x, t, m, \varepsilon)$.

Proof.
1) follows from the formula for $a_1(x, t, m, \varepsilon)$:

$$a_1(x, t, m, \varepsilon) = (1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2} \sum_{r=0}^{(t-|x|)/2\varepsilon} (-1)^r \binom{(t+x-2\varepsilon)/2\varepsilon}{r} \binom{(t-x-2\varepsilon)/2\varepsilon}{r} (m\varepsilon)^{2r+1}.$$

2) is proved using formula for $a_2(x, t, m, \varepsilon)$:

$$a_2(x, t, m, \varepsilon) = (1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2} \sum_{r=1}^{(t-|x|)/2\varepsilon} (-1)^r \binom{(t+x-2\varepsilon)/2\varepsilon}{r} \binom{(t-x-2\varepsilon)/2\varepsilon}{r-1} (m\varepsilon)^{2r}.$$

Main theorem (1)

Theorem

For each $m > 0$ and each point $(x, t) \in \varepsilon\mathbb{Z}^2$ such that $(x + t)/\varepsilon$ is even and $t > |x|$ we have $P(x, t, m, \varepsilon) \neq 0$.

In other words, $P(x, t, m, \varepsilon) \neq 0$ if and only if there exists at least one checker path from $(0, 0)$ to (x, t).

Proof.

Denote $M = \{(x, t) \in \varepsilon\mathbb{Z}^2 \,:\, (x + t)/\varepsilon$ is even,
$t > |x|, P(x, t, m, \varepsilon) = 0\}$.
If $M = \emptyset$, then there is nothing to prove.
Assume that $M \neq \emptyset$.
Among the points of M, select the one with the minimal $t-$coordinate (if there are several such points, select any of them).
Denote by (x_0, t_0) the selected point.

Main theorem (2)

Proof.

For all $t \in \varepsilon\mathbb{Z}_+$ we have
$P(-t+2\varepsilon, t) = m^2\varepsilon^2(1+m^2\varepsilon^2)^{(1-t/\varepsilon)} \neq 0.$
Thus $x_0 \neq -t_0 + 2\varepsilon$.
We have $a_1(-x_0, t_0, m, \varepsilon) = a_1(x_0, t_0, m, \varepsilon) = 0$;
$a_2(2\varepsilon - x_0, t_0, m, \varepsilon) = (t_0 - x_0)\frac{a_2(x_0,t_0,m,\varepsilon)}{t_0+x_0-2\varepsilon} = 0.$
We have

$$a_1(-x_0+\varepsilon, t_0-\varepsilon, m, \varepsilon) = \frac{a_1(-x_0, t_0, m, \varepsilon) - m\varepsilon\, a_2(-x_0+2\varepsilon, t_0, m, \varepsilon)}{\sqrt{1+m^2\varepsilon^2}} = 0;$$

$$a_2(-x_0+\varepsilon, t_0-\varepsilon, m, \varepsilon) = \frac{a_2(-x_0+2\varepsilon, t_0, m, \varepsilon) + m\varepsilon\, a_1(-x_0, t_0, m, \varepsilon)}{\sqrt{1+m^2\varepsilon^2}} = 0.$$

(0,...)		(...,0)	t_0
	(0,0)		t_0-ε
$-x_0$	$-x_0+\varepsilon$	$-x_0+2\varepsilon$	

Figure: The pair in a cell (x, t) is $(a_1(x,t), a_2(x,t))$.

Main theorem (3)

Proof.
Thus $P(-x_0 + \varepsilon, t_0 - \varepsilon, m, \varepsilon) = 0$.
This contradicts to the minimality of t_0, because $(x_0 - \varepsilon + t_0 - \varepsilon)/\varepsilon$ is even and $t_0 - \varepsilon > |x_0 - \varepsilon|$ by the condition $x_0 \neq -t_0 + 2\varepsilon$ above. □

(0,...)		(...,0)	t_0
	(0,0)		t_0-ε
$-x_0$	$-x_0+\varepsilon$	$-x_0+2\varepsilon$	

Figure: The pair in a cell (x, t) is $(a_1(x, t), a_2(x, t))$